12

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12-1 Pyramids and Prisms

The pyramid shape was used by many ancient civilizations. These pyramids were built by Egyptians. They are examples of polyhedra.

A polyhedron is a 3-dimensional object made up of polygonal regions called faces. The sides and vertices of the faces are called edges and vertices of the polyhedron.

Definition 12-1

A polyhedron consists of a finite number of polygonal regions. Each edge of a region is the edge of exactly one other region. If two regions intersect, then they intersect in an edge or a vertex.

This triangular prism is a special kind of polyhedron.

Definition 12-2

A pyramid is a polyhedron in which all faces but one have a vertex in common.

That common vertex is called the vertex of the pyramid, and the face that does not contain the vertex is called the base of the pyramid.
For both pyramids and prisms, the faces that are not bases are called *lateral faces* and the non-base edges are called *lateral edges*. A segment between the bases of a prism perpendicular to the bases is an *altitude*. A segment from the vertex to the base of a pyramid perpendicular to the base is an *altitude*.

A pyramid is *regular* if its base is a regular polygon and its lateral edges are congruent.

A prism is a *right prism* if its lateral edges are perpendicular to its bases.

This theorem states an important characteristic of prisms.

**Theorem 12-1** The lateral edges of a prism are parallel and congruent.
EXERCISES

A.

1. Which one of these figures is not a pyramid? Why?
   a. 
   b. 
   c. 

2. Which one of these figures is a prism? Why?
   a. 
   b. 
   c. 

3. Name five base edges in this pyramid.
4. Name five lateral edges in this pyramid.
5. Identify five lateral faces in this pyramid.
6. Name the base edges in this prism.
7. Name the lateral edges in this prism.
8. Identify the lateral faces in this prism.
9. In any pyramid how do the number of base edges compare to the number of lateral edges?
10. In any prism how do the number of base edges compare to the number of lateral edges?
11. If we view the cube at the right as a prism and we consider $ABCD$ a base, name the second base and the lateral edges.
12. If for the cube at the right we consider $ABFE$ a base, name the second base and the lateral edges.
If the bases of a prism are a parallelogram, the prism is called a parallelepiped. Exercises 13–19 are about a parallelepiped.

13. If $BCGF$ is one base of the prism, name the second base.

14. If $ABFE$ is one base of the prism, name the second base.

For exercises 15–19, $BC = 8$, $AB = 6$, and $BF = 5$. Find each of the following lengths.

15. $AE = \ ?$

16. $EH = \ ?$

17. $CD = \ ?$

18. $DH = \ ?$

19. Name the four diagonals of this parallelepiped.

20. Sketch a pyramid with a quadrilateral base. Hidden edges should be represented by drawing dashed lines. (*Hint:* First draw the base, then choose the vertex, then connect the vertex to the vertices of the base.)

21. Sketch a pyramid with a hexagonal base.

22. Sketch a prism with a hexagonal base.

23. Suppose all faces of a parallelepiped are rectangles and $AB = 15$ cm and $AD = 8$ cm. Show that $AC = 17$.

24. Show that the diagonal $AG$ has length $\sqrt{338}$.

25. Given a parallelepiped with all rectangular faces as shown. If $EH = 10$, $DC = 4$, and $FB = 4$, find the length of diagonal $HB$.

26. Suppose all faces of a parallelepiped are rectangles. If $AC = a$, $AB = b$, and $EC = c$, show that $BE = \sqrt{a^2 + b^2 + c^2}$.

27. In the cube shown, $AB = 5$. Find the length of diagonal $AC$. 

(Exs. 13–19)
28. In this figure the polyhedron with vertices $ABCDP$ is what type of polyhedron?

29. The segments from $P$ to each of the vertices of the cube divide the cube into how many pyramids?

30. Name the base of each of the pyramids counted in Exercise 29.

31. Suppose a cube is sliced by a plane $ACF$ as shown, forming a pyramid $ABCF$. Explain why this pyramid is a regular pyramid. (Recall that the edges of a cube are all the same length.)

32. What is the base of the regular pyramid that has been cut from the cube?

33. How many regular pyramids like $ABCF$ can be sliced from the cube? Name them.

34. After four of the pyramids like $ABCF$ have been removed from the cube shown, the polyhedron $ACHF$ remains. Name all the faces of $ACHF$, and explain why they are all equilateral triangles.

---

**Activity**

Outlined below is a method for constructing a triangular-based pyramid from a sealed envelope. Complete the construction.

1. Construct point $C$ so that $\triangle ABC$ is an equilateral triangle.

2. Cut along $DE$, through $C$, and parallel to $AB$.

3. Fold along $AC$ and $BC$ back and forth in both directions.

4. Let $C'$ be the point on the reverse side corresponding to point $C$.

5. Open and pinch the envelope so that points $D$ and $E$ are joined and $C$ and $C'$ are separated. Tape along segment $CC'$ and the pyramid is complete.

Construct a solid using this method.
35. In this parallelepiped explain why point $O$ is the midpoint of $AG$ and $BH$.

36. The segments $CE$ and $AG$ are the diagonals of what parallelogram?

37. Explain why point $O$ is the midpoint of $CE$.

38. A school hall 9 feet high and 9 feet wide turns a corner as shown. (This can be viewed as a pair of intersecting parallelepipeds with rectangular faces.) Can a pole 12 feet long (like one used in pole vaulting) be carried down the hall and around the corner? Explain.

39. A top view of the hall is shown. If $AB \parallel CD$, verify that $CD = 18\sqrt{2}$.

40. Could a pole slightly longer than $18\sqrt{2}$ feet be carried down the hall and around the corner? Explain.

---

**PROBLEM SOLVING**

A box with dimensions 2 by 4 by 8 can be wrapped in either of two methods. How much ribbon is required in each case? *(Hint: Think of cutting the box open to calculate the ribbon length for method 2. In the diagram some faces are shown twice.)*
12-2 Surface Area of Prisms and Pyramids

Professional interior decorators and do-it-yourself remodelers need to determine the quantity of materials required to decorate surfaces. Sometimes familiar objects, such as end tables or cabinets, have prism shapes. It is often necessary to calculate surface areas of these shapes.

The areas of the surfaces of prisms and pyramids can be found using the following rule:

\[
\text{Surface Area} = \text{Sum of the areas of the lateral faces} + \text{Area of the bases}
\]

Consider a prism with altitude \( h \), rectangular lateral faces, and pentagonal bases.

If the area of each base is \( B \) and the base edges have lengths \( e_1, e_2, e_3, e_4, \) and \( e_5 \), then

\[
\text{Area of lateral faces} = e_1h + e_2h + e_3h + e_4h + e_5h
\]

\[
= h(e_1 + e_2 + e_3 + e_4 + e_5)
\]

\[
=.hp, \text{ where } p \text{ is the perimeter of the base.}
\]

**Theorem 12-2** Given a prism with rectangular lateral faces. If the altitude of the prism is \( h \) and the bases have area \( B \) and perimeter \( p \), then the surface area \( S \) is found by the formula \( S = hp + 2B \).
The surface area of a pyramid is equal to the sum of the areas of the lateral faces plus the area of the base.

Consider a regular pyramid with pentagon base, slant height \( l \), and base edges with lengths \( e_1 = e_2 = e_3 = e_4 = e_5 \).

Sum of areas of the lateral faces

\[
= \frac{1}{2} e_1 l + \frac{1}{2} e_2 l + \frac{1}{2} e_3 l + \frac{1}{2} e_4 l + \frac{1}{2} e_5 l \\
= \frac{1}{2} l(e_1 + e_2 + e_3 + e_4 + e_5) \\
= \frac{1}{2} lp, \text{ where } p \text{ is the perimeter of the base.}
\]

This information is summarized in the following theorem.

**Theorem 12–3** Given a regular pyramid with slant height \( l \) and a base with area \( B \) and perimeter \( p \), the surface area \( S \) is found by the formula

\[
S = \frac{1}{2} lp + B.
\]

**APPLICATION**

Sometimes formulas need to be adapted to be applied to a given physical object. For example, consider the plumb (a weight used in construction) shown here. Its shape is that of a hexagonal prism with a regular hexagon base attached to a hexagonal pyramid at the bottom. A manufacturer needs to know its surface area.

From the drawing we can calculate that the area of the base is \( 6 \sqrt{3} \) cm\(^2 \). Using Theorems 12–2 and 12–3, we find the surface area of the prism and pyramid:

Surface area of prism = \( (12)(8) \) cm\(^2 \) + \( 12 \sqrt{3} \) cm\(^2 \),  
Surface area of pyramid = \( \frac{1}{2}(5)(12) \) cm\(^2 \) + \( 6 \sqrt{3} \) cm\(^2 \).

But one base of the prism and the base of the pyramid are joined. Because neither base is part of the surface of the plumb, we must subtract the base area twice. Therefore,

Surface area of plumb

\[
= (12)(8) + 12 \sqrt{3} + \frac{1}{2}(5)(12) + 6 \sqrt{3} - 2(6 \sqrt{3}) \\
= (126 + 6 \sqrt{3})\text{cm}^2.
\]
EXERCISES

A.

For exercises 1 and 2, select the correct formula for finding the surface area. \( p \) is the perimeter of the base, \( h \) is the altitude, \( l \) is the slant height, and \( B \) is the area of the base(s).

1. a. \( S = \frac{1}{2} ph + 2B \)
   b. \( S = ph + B \)
   c. \( S = ph + 2B \)
   d. \( S = pl + 2B \)

2. a. \( S = \frac{1}{2} pl + B \)
   b. \( S = pl + B \)
   c. \( S = \frac{1}{2} pl + 2B \)
   d. \( S = \frac{1}{2} pl + B \)

For exercises 3–5, find the surface area of these right prisms and a regular pyramid.

3. 4.

6. Find the surface area of a box with no top that is 5 units long, 3 units wide, and 2 units high.

7. Find the surface area of a right prism with equilateral triangle bases if all edges are 2 units long.

B.

Find the surface area of these regular pyramids.

8. 9. 10.

Activity

A deltahedron is a polyhedron with triangular faces. Make models of the deltahedra by gluing toothpicks together. Which of the deltahedra are pyramids?
11. The surface of a square-based prism is $360 \text{ cm}^2$ and the height is twice the length of a base edge. What are the lengths of the edges of this prism?

12. The surface area of a square-based pyramid is $48 \text{ cm}^2$. If the slant height is equal to the base edge, what is the area of the base?

13. What is the length of the altitude of the pyramid in exercise 12?

14. If the length of each edge of a prism is doubled, how does the surface area change?

15. Square cake pans $20 \text{ cm}$ on an edge and $6 \text{ cm}$ deep are to be coated on the inside with a non-stick material. If the amount of non-stick material available covers $100 \text{ square meters}$, how many pans can be coated?

16. A large container shaped like a regular pyramid has an open top. The top is a regular hexagon with dimensions as shown. If one hundred of these containers are to be painted, both inside and out, with a paint that covers $450 \text{ square feet}$ per gallon, how many gallons of paint are needed?

**Problem Solving**

In the figure at the right 14 cubes have been stacked to form a solid whose surface area (including the base) is 42 units.

1. How can the surface area be changed to 44 units by moving only one block?

2. How can the surface area be changed to 40 units by moving only one block?
Volume of Prisms

A civil engineer estimates construction costs. At this highway construction site the engineer determines the amount of earth material to be moved in reshaping the terrain by calculating the volume.

There are postulates that characterize the concept of volume. We shall study these postulates in this lesson.

Intuitively, we think of volume as a measure of the amount of space occupied by a solid.

We begin our study of volume by considering a solid commonly referred to as a “box” shape which we define to be a rectangular solid.

A rectangular solid has length, width, and height.

Volume Postulate
Each solid is assigned a unique positive number called its volume.

Definition 12-4
A rectangular solid is a prism with rectangular bases whose lateral edges are perpendicular to the bases.

Rectangular Solid Volume Postulate
The volume of a rectangular solid is equal to the product of its length $l$, width $w$, and height $h$.

Example
Find the volume $(V)$ of a box $8 \text{ cm} \times 4 \text{ cm} \times 2 \text{ cm}$.

$$V = 2 \text{ cm} \times 4 \text{ cm} \times 8 \text{ cm} = 64 \text{ cm}^3$$

(Read 64 cubic centimeters.)

This is equivalent to counting the number of cubes $1 \text{ cm}$ on a side that fit into the box.

Volume Addition Postulate
If a solid is the union of two solids that have no common interior points, then its volume is the sum of the volumes of the two solids.
Imagine that a rectangular solid has been sliced into thin layers and that the layers shift to form irregular-shaped solids. The volume of the solid remains the same.

Similarly, suppose that two solids can be sliced into thin layers so that the tops of corresponding layers have equal areas. Intuition suggests that the volumes of the two solids are equal.

**Definition 12-5**

A cross section of a solid is a region common to the solid and a plane that intersects the solid.

These examples lead to a postulate known as Cavalieri’s Principle, named after the seventeenth century Italian mathematician Bonaventura Cavalieri (1598–1647).

**Cavalieri’s Postulate**

Let $S$ and $T$ be two solids and $X$ be a plane. If every plane parallel to $X$ that intersects $S$ or $T$ intersects both $S$ and $T$ in a cross section having the same area, then

$$\text{Volume } S = \text{Volume } T.$$

The postulates from this lesson can be combined to prove the following theorem.

**Theorem 12-4** The volume of any prism is the product of the length of an altitude and the area of the base.
EXERCISES

A.

Find the volume of each of these boxes.

1. \[ \text{3 cm} \times \text{2 cm} \times \text{4 cm} \]

2. \[ \text{3 cm} \times \text{4 cm} \times \text{5 cm} \]

3. \[ \text{3.5 cm} \times \text{2.6 cm} \times \text{7.1 cm} \]

4. How many cubes 1 cm on a side will fit into the box in exercise 1?

5. How many cubes 2 cm on a side will fit into the box in exercise 1?

6. How many cubes 1 cm on a side will fit into the box in exercise 2?

7. How many cubes 1 cm on a side will fit into the box in exercise 3?

8. How many cubic inches are there in one cubic foot?

Find the volume of the prisms in exercises 9–11.

9. \[ \text{8 cm} \times \text{5 cm} \times \text{3 cm} \]

10. \[ \text{4 cm} \times \text{5 cm} \times \text{9 cm} \]

11. \[ \text{regular hexagon base} \]

B.

12. If the area of the base of a prism is doubled and the height remains the same, how much does the volume increase?

Activity

Draw two copies of this figure and make two solid figures.

Fit those two solids together to form a prism.

(When drawing this figure, make sure that polygon 1 is a regular hexagon and polygons 2 are 45-45-90 triangles.)
13. If the lengths of all sides of a box are doubled, how much is the volume increased?

14. Silver ingots are molded in bars shaped as shown. The ends are parallel isosceles trapezoids. What is the volume?

15. A rectangular container is 5 cm wide and 12 cm long and contains water to a depth of 7 cm. A stone is placed in the water and the water rises 1.7 cm. What is the volume of the stone?

16. A heating engineer needs to find the volume of a building in order to design its heating system. Find the volume of the building pictured.

17. If a rectangular container with a square base is 2 feet high and has a volume of 50 cubic feet, find the length and width of the base.

18. Suppose the area of a base of a prism is $x$ square feet and its height is $2x$ feet. If the volume of the prism is 54 cubic feet, how tall is the prism?

19. An engineer's plan shows a canal with a trapezoid cross section, 8 feet deep and 14 feet across at the bottom, and walls sloping outward at an angle of $45^\circ$. The canal is 620 feet long. A contractor estimates that it will cost $1.50 per cubic yard to excavate the canal. If the contractor adds 10% for profit, what should the bid be?

20. A concrete retaining wall is 80 feet long with ends shaped as shown. How many cubic yards of concrete are used in constructing this wall?

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**PROBLEM SOLVING**

How many cubic yards of concrete are needed for the steps shown in the diagram?
12–4 Volume of Pyramids

There are many occasions when it is necessary to find the volume of an object that is not a rectangular solid or a prism. This sketch shows a dairy product container that is a triangular-based pyramid. Knowing the volume of the container is essential to the dairy company.

Theorem 12–7 gives the formula for the volume of a pyramid. First we must state two other theorems. These theorems are used in the development of the formula.

Theorem 12–5
Given a pyramid with base $B$ and altitude $h$. If $A$ is a cross section parallel to the base and the distance from the vertex to the cross section is $K$, then

$$\frac{\text{area } A}{\text{area } B} = \left(\frac{K}{h}\right)^2$$

Theorem 12–6
Two pyramids with equal altitudes and bases of equal area have the same volume.

Suppose we want to find the volume of pyramid $WXYZ$. We shall do this by finding the volume of a right pyramid $ABCD$ with base area equal to the area of base $\triangle XYZ$ and altitude equal to $h$. Use Theorem 12–6.
Next consider a right prism with the same base and altitude as pyramid $ABCD$.

Make the following two cuts in the prism:

1. Cut from $A$ through $BD$.
2. Cut from $A$ through $ED$.

Then

1. Volume of $ABCD = \text{Volume } ADEF$. Bases $\triangle BCD$ and $\triangle AEF$ have equal area, and altitudes $AC$ and $DF$ are equal in length. Therefore Theorem 12–6 implies the equality of volume.

2. Volume of $ADEF = \text{Volume of } ABDE$. These two pyramids have bases $\triangle BDE$ and $\triangle FDE$ that have equal areas since they are each half of rectangle $BDFE$. The altitudes for both pyramids are formed by a perpendicular segment from $A$ to the opposite base and so the altitudes are equal. Theorem 12–6 states that the volumes are equal.

Therefore, the volume of pyramid $ABCD = \frac{1}{3}$ the volume of prism $ABCDEF$

$$= \frac{1}{3}hB.$$  

**Theorem 12–7** Given a pyramid with altitude $h$ and base area $B$, the volume is found by the formula $V = \frac{1}{3}hB$. 

EXERCISES

A.

Answer exercises 1–5 true or false.

1. If two pyramids of equal altitudes have congruent bases, then their volumes are equal.
2. If two pyramids have equal volumes, then their altitudes are necessarily equal.
3. If two pyramids have equal volumes and equal altitudes, then their bases are necessarily congruent.
4. If two pyramids have equal volumes and bases equal in area, then their altitudes are necessarily equal.
5. A pyramid with a square base can never have a volume equal to a pyramid with a triangular base.

Find the volume of these pyramids.

6. 

```
18
area of base = 51
```

7. 

```
5
4
area: base = 65
```

B.

In exercises 9–11, find the volume of the given regular pyramid.

9. 

```
22
25
```

10. 

```
24
35
```

11. 

```
27
16
```

Activity

Draw an enlargement of this figure, cut out two copies, fold, and tape together to make two polyhedra. Can you put the two polyhedra together to form a pyramid? (When drawing this figure make sure polygon 1 is a square and that polygons 2 and 3 taken together form a regular hexagon.)
12. The bases of these two pyramids have equal area. How do their volumes compare?

13. These two pyramids have equal altitudes and square bases. How do their volumes compare?

14. What is the area of the cross section $A$ of this pyramid?

15. What is the volume of the shaded portion of this pyramid?

16. What is the volume of a regular octahedron whose edge lengths are 3?

17. A water retention basin is located along side of a parking lot. The basin begins at point $A$ and deepens as it widens. The top edge $BC$ of the deep end is 8 meters wide. Point $B$ is 40 meters from $A$. The deepest point $D$ is 1.5 meters below $BC$. There is a drain at $D$ that drains 50 liters per minute. Given that 1 m$^3$ holds 1000 liters, how many hours will it take for the basin to drain when it is full? (Assume that $ABCD$ is a pyramid.)

**Problem Solving**

Many blueprint drawings show the top, side, and front views of objects. Notice the use of dashed lines to show cuts hidden from view.

Sketch two "cut blocks" that have the same top and front views but different side views.
12-5 Surface Area and Volume of Cylinders

Many familiar objects are examples of the cylindrical shape. These pictures show some of them. In this lesson we define a circular cylinder and describe formulas for calculating its surface area and volume.

A cylinder is like a prism in that it has congruent bases in a pair of parallel planes. The bases are congruent circular regions.

The segment joining the centers of the two bases is called the axis of the cylinder. A cylinder is a right cylinder if its axis is perpendicular to the bases. The height of the cylinder is the length of the axis.

A cylinder can be thought of as a prism with an infinite number of sides. The lateral surface and the circumference of the bases of a cylinder correspond to the lateral faces and perimeter, respectively, of a prism.
The next two theorems describe the surface area and volume of a right circular cylinder.

**Theorem 12–8** Given a right circular cylinder with altitude $h$. If the circumference of the base is $C$, and the area of the base is $B$, then the surface area $S$ is found by the formula

$$S = Ch + 2B = 2\pi rh + 2\pi r^2.$$  

**Theorem 12–9** Given a right circular cylinder with base area $B$ and height $h$, the volume is found by the formula

$$V = Bh = \pi r^2 h.$$  

**Example 1**

A container in the shape of a right circular cylinder is 35 cm tall and 16 cm in diameter. Find the surface area and the volume of this container.

$$S = 2\pi(8) \cdot 35 + 2\pi(8)^2$$
$$= 560\pi + 128\pi$$
$$= 688\pi \text{ cm}^2$$

$$V = \pi(8)^2 \cdot 35$$
$$= 2240\pi \text{ cm}^3$$

**Example 2**

If the radius and height of a cylinder double, how much do the surface area and volume change?

$$S(\text{large cylinder}) = 2\pi(2r)(2h) + 2\pi(2r)^2$$
$$= 4(2\pi rh) + 4(2\pi r^2)$$
$$= 4S \text{ (small cylinder)}$$

$$V(\text{large cylinder}) = \pi(2r)^2(2h)$$
$$= 8\pi r^2 h$$
$$= 8V \text{ (small cylinder)}$$
EXERCISES

A.

Find the surface area and volume of the cylinders in exercises 1–3.

1. 

2. 

3. 

4. A cylindrical tank is 17 feet high and has a base radius of 10 feet. How many cubic feet are contained in the tank?

5. How many cubic yards are contained in the tank in exercise 4?

B.

6. A marble column in the shape of a right circular cylinder is 9 meters high and 80 cm in diameter. If 1 m³ of marble weighs 300 kg, find the weight of the column.

7. The volume of a right circular cylinder is 972 cm³. If the height is 12 cm, what is the radius of the base?

8. Two right circular cylinders with the same height have radii with a ratio of 2:1. What is the ratio of the volumes of the two cylinders?

Activity

Make a solid out of clay, wood, or other material that completely plugs up each of these holes and can be pushed through each with no change in shape.
9. A cylindrical hole with diameter 8 inches is cut through a cube. The edge of the cube is also 8 inches. Find the volume and the surface area of this hollow solid.

10. A $4 \times 7$ rectangle is rotated about the long side to generate a cylinder and is also rotated about the short side to generate a cylinder. What is the ratio of the volumes of these cylinders?

11. A case is packed with six cylindrical tin cans. What is the ratio of the volume of the box to the combined volumes of the tin cans?

12. Find the surface area and the volume of this solid steel casting.

**Problem Solving**

Find the volume and surface area of this bearing washer.
12-6 Surface Area and Volume of Cones

The cone shape is often found in the physical world in combination with the cylinder shape. For example, the top of storage bins, the tip of a pencil, and the tip of a nail can all be viewed as a cone mounted on a cylinder. A child might make a sand castle by combining these shapes.

The figure on the right is a right circular cone. It has a circular base and a vertex. Its axis is the segment joining the vertex to the center of the base. The cone is called a right cone since the axis is perpendicular to the base.

A cone can be thought of as a pyramid with an infinite number of lateral faces. The lateral surface of a cone corresponds to the lateral faces of a pyramid. The slant height \( l \) of a cone corresponds to the slant height \( l \) of a pyramid, and the circumference \( C \) of the base of a cone corresponds to the perimeter \( p \) of the base of a pyramid.
The next theorem describes the surface area of a cone.

**Theorem 12-10** Given a right circular cone with slant height \( l \). If the circumference of the base is \( C \), and the area of the base is \( B \), then the surface area \( S \) is given by the formula

\[
S = \frac{1}{2}CL + B = \pi rl + \pi r^2.
\]

The formula for volume given in this next theorem is similar to the formula for the volume of a pyramid.

**Theorem 12-11** Given a right circular cone with height \( h \) and base area \( B \), the volume is found by the formula

\[
V = \frac{1}{3}hB = \frac{1}{3}\pi r^2h.
\]

**Example**

A right circular cone has height 15 and base radius of 8. Find the slant height, the surface area, and the volume.

a. slant height:

\[
l^2 = 64 + 225 = 289 \quad l = 17
\]

b. \( S = \pi(8)(17) + \pi 64 = 200\pi \)

c. \( V = \frac{1}{3}\pi(8)^215 = 320\pi \)
EXERCISES

A.

For exercises 1-3, find the volume and surface area of the right cones shown.

1. 

2. 

3. 

4. The radius of a cone is 5 cm and its altitude is 12 cm. Find its surface area and volume.

5. If the volume of a cone is $72\pi$, find its height and radius if they are equal.

B.

6. A container is composed of a right circular cylinder of diameter 4 cm and height 8 cm surmounted by a cone of height 6 cm. Find the volume of this container.

7. Find the surface area of this container.

8. Find the volume of this toy top.

9. Find the surface area of this top.

10. A sand pile is shaped like a cone as shown. How many cubic yards of sand are in this pile? (Large quantities of sand are purchased by the cubic yard.)

Activity

Make, or otherwise obtain, models of a cone and a cylinder with an open top that have equal radii and height.

Fill the cone to the top with sand and dump the sand into the cylinder.

How many cones of sand are required to fill the cylinder?
11. How many cubic inches of lead are there in the sharpened tip of this pencil? (Use a calculator.)

12. This solid is formed by cutting a cone with a slice parallel to the base of the cone. Find its volume and surface area. (Hint: Use similar triangles to find the height of the original cone.)

13. The legs of a right triangle have lengths of 2 and 3. Cones are formed by revolving the triangles about the shorter and longer sides. Find the ratio of volumes and the ratio of surface areas of these two solids.

14. This solid is formed by cutting a cone with a slice parallel to the base, and then boring a cone-shaped cut into the resulting solid. Find the volume of this solid.

**PROBLEM SOLVING**

Match the object in Set 1 with its correct view (top or side) from Set 2.
12-7 Surface Area and Volume of Spheres

In this lesson we shall study formulas for the volume and surface area of a sphere.

The given point $O$ is called the center of the sphere. A radius of a sphere is a segment determined by the center and a point on the sphere. The intersection of a sphere and a plane containing the center of the sphere is called a great circle of the sphere.

Definition 12-6

A sphere is the set of all points that are a given distance from a given point.

The explanation of the formula in Theorem 12-12 is based upon a comparison between a sphere and a cylinder that has had a double cone carved out of it. The radii of the sphere and cylinder are equal in length. The height of the cylinder is twice the radius.

Theorem 12-12

Given a sphere of radius $r$, the volume is found by the formula

$$V = \frac{4}{3}\pi r^3.$$
Consider a cross section of both the sphere and the carved out cylinder that is a distance $b$ from the center of the sphere. We conclude from the Pythagorean Theorem that the distance $a$ in the figure is $a^2 = r^2 - b^2$.

The triangle in red is an isosceles triangle.

Compare the areas of the two cross sections.

Area of $A = \pi a^2$

Area of $B = \pi r^2 - \pi b^2 = \pi(r^2 - b^2) = \pi a^2$

Since the areas of each cross section are equal, we conclude from Cavalieri’s Principle (page 445) that the volume of the sphere $A$ is equal to the volume of the solid $B$ from which two cones are carved.

The volume of solid $B$ can be calculated as:

$$\pi r^2(2r) - 2\left(\frac{1}{3}\pi r^2\right)(r) = 2\pi r^3 - \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3.$$

Then the volume of sphere $A$ with radius $r$ is also $\frac{4}{3}\pi r^3$.

Theorem 12-13 gives a formula for the surface area of a sphere.

**Theorem 12-13** Given a sphere of radius $r$, the surface area $S$ is found by the formula

$$S = 4\pi r^2.$$
EXERCISES

A.
1. Find the volume of a sphere of radius 9 cm.
2. Find the surface area of a sphere of radius 9 cm.
3. Find the volume of a sphere of radius 2π.
4. Find the surface area of a sphere of radius 2π.
5. If the surface area of a sphere is 36π, find the radius.
6. If the volume of a sphere is 36π, find the radius.
7. If the surface area of a sphere is 8π, find the radius.
8. If the volume of a sphere is 4π√3, find the radius.

B.
For exercises 9 and 10 assume the solid is a right cylinder capped by two hemispheres.
9. Find the volume of the solid shown.
10. Find the surface area of the solid shown.
11. Find the volume of a sphere whose surface area is 144π square units.
12. Find the surface area of a sphere whose volume is 36π cubic units.
13. If the number of square feet of surface area of a sphere is equal to the number of cubic feet of volume of the sphere, what is the radius of the sphere?

Activity

Obtain a hemispherical container and a cylindrical container whose base diameter and height equal the diameter of the sphere.

Using sand or other material, measure how many cupfuls are required to fill the cylinder.
14. The radius of one sphere is twice the radius of another sphere. What are the ratios of their volumes and surface areas?

15. A sphere is inscribed in a cylinder. Show that the surface area of the sphere is equal to the lateral surface area of the cylinder.

16. A spherical tank whose radius to the outer surface is 15 feet is made of steel \( \frac{1}{2} \) inch thick. How many cubic feet of steel are used in the construction of this tank?

17. How does the volume of a cone whose height is twice its radius compare to the volume of a sphere whose radius equals the radius of the cone base?

18. The shape of the Earth is an oblate spheroid and not a perfect sphere. Determine the average radius of the Earth given that the Polar radius is 6357 km and the Equatorial radius is 6378 km. Assume that the Earth is a perfect sphere and determine its volume and surface area.

**Problem Solving**

If a sphere is sliced with a plane a portion of a sphere with a circular base is formed as shown.

Fact: If the radius of this circular base is \( r \), and its height is \( h \), then the volume of this solid cap is

\[ V = \frac{1}{2} \pi r^2 h + \frac{1}{6} \pi h^3. \]

If a hole 3 cm in radius is drilled through the center of a sphere of radius 9 cm, find the volume of this bead-like solid.
12-8 Regular Polyhedra

Some naturally occurring minerals and skeletons of tiny sea creatures are models of solids that we study in this lesson. These solids are called polyhedra.

Definition 12-7

A regular polyhedron is a polyhedron whose faces are all regular polygons with the same number of edges and whose vertices are each surrounded by the same number of faces.

A cube is an example of a regular polyhedron. The method of constructing a cube described here by forming a "roof" is instructive in our analysis of the theorem below.

Surround a vertex (V) with three squares. Fold, join A and B to form a 3-dimensional "roof." Join two "roofs" to form a cube.

Theorem 12-14 There are exactly five regular polyhedra that are convex solids.
This table summarizes the five convex regular polyhedra. There are only five types of "roofs" that can be constructed with regular polygons. Each one results in a regular polyhedron.

<table>
<thead>
<tr>
<th>Polygon Face</th>
<th>Number of faces at a vertex ($V$)</th>
<th>Join $\overline{AV}$ and $\overline{BV}$ to form a 3 dimensional &quot;roof.&quot;</th>
<th>The &quot;roof&quot; fits over each vertex of a complete regular polyhedron.</th>
</tr>
</thead>
<tbody>
<tr>
<td>equilateral triangle</td>
<td><img src="image1" alt="Image" /></td>
<td><img src="image2" alt="Image" /></td>
<td><img src="image3" alt="Image" /></td>
</tr>
<tr>
<td>equilateral triangle</td>
<td><img src="image4" alt="Image" /></td>
<td><img src="image5" alt="Image" /></td>
<td><img src="image6" alt="Image" /></td>
</tr>
<tr>
<td>equilateral triangle</td>
<td><img src="image7" alt="Image" /></td>
<td><img src="image8" alt="Image" /></td>
<td><img src="image9" alt="Image" /></td>
</tr>
<tr>
<td>square</td>
<td><img src="image10" alt="Image" /></td>
<td><img src="image11" alt="Image" /></td>
<td><img src="image12" alt="Image" /></td>
</tr>
<tr>
<td>regular pentagon</td>
<td><img src="image13" alt="Image" /></td>
<td><img src="image14" alt="Image" /></td>
<td><img src="image15" alt="Image" /></td>
</tr>
</tbody>
</table>

The prefix used in the name of each regular polyhedron tells the number of faces for that polyhedron. This information is summarized in this table.

<table>
<thead>
<tr>
<th>Polyhedron Name</th>
<th>Prefix and Prefix Meaning</th>
<th>Number of Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>tetrahedron</td>
<td>tetra—4</td>
<td>4</td>
</tr>
<tr>
<td>cube (hexahedron)</td>
<td>hexa—6</td>
<td>6</td>
</tr>
<tr>
<td>octahedron</td>
<td>octa—8</td>
<td>8</td>
</tr>
<tr>
<td>dodecahedron</td>
<td>dodeca—12</td>
<td>12</td>
</tr>
<tr>
<td>icosahedron</td>
<td>icosa—20</td>
<td>20</td>
</tr>
</tbody>
</table>
EXERCISES

A.

1. Name the three regular polyhedra with equilateral triangular faces.
2. Which regular polyhedron has 20 faces?
3. Why can a regular hexagon not be the face of a regular polyhedron?
4. Why can there not be six faces at a vertex in a regular polyhedron?
5. Which regular polyhedron is a pyramid?
6. Which regular polyhedron is a prism?

B.

Use stiff paper to construct a model of each regular polyhedron. The patterns shown below should be enlarged. Cut on solid lines, fold on dotted lines.

7. Tetrahedron
8. Cube
9. Octahedron

Activity

1. Cut from posterboard two large patterns like the one above.
2. Crease sharply along edge ABCDE.
3. Place one pattern upon the other one rotated 36°. While holding the patterns in place, weave an elastic band alternately above and below the corners.
4. When you raise your hand, you will see a dodecahedron.
C. The number of faces \( F \), the number of edges \( E \), and the number of vertices \( V \) of a polyhedron satisfy one of the formulas below.

12. Which one formula below is satisfied by the regular tetrahedron and regular octahedron?
   \[ a. \ F + E + V = 26 \quad b. \ F - V + E = 10 \]
   \[ c. \ F - E + V = 2 \quad d. \ F - E + V = 0 \]
   The correct answer to this exercise is called **Euler's Formula**.

For which of these polyhedra does Euler's Formula hold?

13.  
   ![Polyhedron](image)

14.  
   ![Polyhedron](image)

15.  
   ![Polyhedron](image)

---

**PROBLEM SOLVING**

Let \( F \) be the number of faces of a polyhedron, \( E \) be the number of edges of a polyhedron, and \( V \) be the number of vertices of a polyhedron. \( F \times \) (number of edges per face) = \( 2E \) because each edge of a polyhedron is an edge of two faces.

1. How many edges does a regular octahedron possess?
2. How many edges does a regular dodecahedron possess?
3. How many edges does a regular icosahedron possess?
4. In a regular dodecahedron \( F \times \) (number of vertices/face) = \( 3V \) because each vertex of a regular dodecahedron is a vertex of 3 faces. How many vertices does a regular dodecahedron possess?
Important Ideas—Chapter 12

Terms

Polyhedron (p. 434)
Pyramid (p. 434)
Prism (p. 435)
Rectangular solid (p. 444)
Cross section (p. 445)
Circular cylinder (p. 452)
Right circular cone (p. 456)
Sphere (p. 460)
Regular polyhedron (p. 464)

Postulates

Volume Postulate (p. 444)
Rectangular Solid Volume Postulate (p. 444)
Volume Addition Postulate (p. 445)
Cavalieri’s Postulate (p. 445)

Theorems

12-1 The lateral edges of a prism are parallel and congruent.
12-2 Given a prism with rectangular lateral faces. If the altitude of the prism is \( h \) and the bases have area \( B \) and perimeter \( p \), then the surface area \( S \) is found by the formula \( S = hp + 2B \).
12-3 Given a regular pyramid with slant height \( l \) and a base with area \( B \) and perimeter \( p \), the surface area \( S \) is found by the formula \( S = \frac{1}{2}lp + B \).
12-4 The volume of any prism is the product of the altitude and the area of the base.
12-5 Given a pyramid with base \( B \) and altitude \( h \). If \( A \) is a cross section parallel to the base and the distance from the vertex to the cross section is \( K \), then
\[
\frac{\text{area } A}{\text{area } B} = \left( \frac{K}{h} \right)^2.
\]
12-6 Two pyramids with equal altitudes and bases of equal area have the same volume.
12-7 Given a pyramid with altitude \( h \) and base area \( B \), the volume is found by the formula \( V = \frac{1}{3}hB \).
12-8 Given a right circular cylinder with altitude \( h \). If the circumference of the base is \( C \), and the area of the base is \( B \), then the surface area \( S \) is found by the formula \( S = Ch + 2B = 2\pi rh + 2\pi r^2 \).
12-9 Given a right circular cylinder with base area \( B \) and height \( h \), the volume is found by the formula \( V = Bh = \pi r^2h \).
12-10 Given a right circular cone with slant height \( l \). If the circumference of the base is \( C \), and the area of the base is \( B \), then the surface area \( S \) is given by the formula \( S = \frac{1}{2}Cl + B = \pi r l + \pi r^2 \).
12-11 Given a right circular cone with height \( h \) and base area \( B \), the volume is found by the formula \( V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2h \).
12-12 Given a sphere of radius \( r \), the volume is found by the formula \( V = \frac{4}{3}\pi r^3 \).
12-13 Given a sphere of radius \( r \), the surface area \( S \) is found by the formula \( S = 4\pi r^2 \).
12-14 There are exactly five regular polyhedra that are convex solids.
Companies that design and build computers have become a vital part of the economy. The computer revolution has caused many new companies to grow, literally from the inventors’ garages. Numerous companies offer personal computers, accessories, and software (the programs that make the computers run). But it is people who must invent, design, test, refine, and market this equipment. These pages show a few of the careers centered in the computer industry.

A computer engineer uses a small computer to test circuits in a mainframe computer.
A prototype is a handmade sample showing how all parts of a new computer fit together. These technicians discuss a problem with a cable prototype.

Testing is a continuous activity.

Industrial designers work to make the computer system attractive and functional. Pleasing shapes, balance between horizontal and vertical, and pleasing ratios of width to height are geometric factors to consider.

Customer service personnel make sure the computer operates correctly.
Computers are used in many industries for many different purposes. Often they monitor the work of other machines. This insures accuracy in manufacture, especially in repetitive jobs. Computers also control machines to achieve critical measurements and exact fit. Although some jobs have been taken over by computers, many others have been created or enriched by computer technology. For people in these industries, geometric ideas, concepts, and skills are important in their training.

2 TV control room operators must view scenes from a variety of angles and select the one most appropriate for the situation.

3 Computerized drafting machines can draw simultaneously in four colors. Geometric visualization skills are necessary for the programmer.
4 Microfilm, viewed through microfiche viewers, is used to catalog parts lists and show three-dimensional drawings of parts.

5 Pocket computers help in measurement.

6 Air traffic controllers use concentric circles, angles, and distance to determine plane location near an airport.

7 A computer drawing of a DNA molecule aids in biological research.

8 Special geometric figures on a bank credit card enable the automatic teller to identify the customer and complete the transaction.

9 The designer of a printed page uses a geometric "sixth sense" to determine the proper balance between type sizes and styles.

10 Bar code scanners give customers better information and help store managers control the inventory.

11 Microcomputers are used for many small business tasks.
Trains, buildings, bridges, and other human creations employ a wide variety of geometric ideas for their design and construction.

A printed “hardcopy” uses different colors to show different automotive systems. These three-dimensional diagrams help mechanic trainees visualize the systems.
Chapter 12—Review

1. Find the surface area and volume of a cube in which each edge is 4 cm.

2. Find the surface area of the regular pyramid and regular prism shown.
   a. 
   
   $AB = 10 \text{ cm}, \ CD = 6 \text{ cm}$
   
   b. 
   
   $BC = 6 \text{ cm}, \ CD = 2 \text{ cm}$

3. Find the volumes of the regular pyramid and regular prism shown.
   a. 
   
   $VM = 8 \text{ cm}, \ AB = 5 \text{ cm}$
   
   b. 
   
   $PQ = 8 \text{ cm}, \ CD = 4 \text{ cm}$

4. How many square centimeters of paper would it take for a label on a cylindrical can 10 cm high and with a circular base of 6 cm in diameter?

5. Find the volume of the cylindrical can described in exercise 4.

6. The volume of a right circular cylinder is $160\pi$. If the height is 10, what is the diameter of the base?

7. Find the volume of the circular cone, where $PA = 12 \text{ cm}$ and $AB = 3 \text{ cm}$.

8. If the height of a right circular cone is doubled, how is the volume affected?

9. A sphere has a volume of $36\pi \text{ cm}^3$. What is its radius?

10. Find the surface area of a sphere with radius 10 cm.
Chapter 12—Test

1. How many cubes with an edge of 2 cm can fit into a box with dimensions 3 cm \( \times \) 10 cm \( \times \) 16 cm?

2. Find the diagonal of a cube if each edge is 1 cm long.

3. Find the surface area of the regular pyramid and right circular cylinder shown.

4. Find the volume of the prism and trapezoid-based pyramid shown.

5. How many cubic centimeters of liquid could be in a right circular cone if its height is 8 cm and its base has a radius of 3 cm?

6. Find the surface area of a right circular cone if the base has a radius of 2 cm and its altitude is 6 cm.

7. Find the volume of the cone described in exercise 6.

8. Find the volume of a sphere with a radius of 3 cm.

9. Find the radius of a sphere if its volume is \( 288\pi \) cm\(^3\).

10. How is the surface area of a sphere affected if the diameter is doubled?

11. How is the volume of a sphere affected if the radius is doubled?
Make an Accurate Drawing

Sometimes an answer to a problem can be found by making an accurate drawing or a scale drawing. Study the example below in which a scale drawing is used to solve a problem.

Example

An 8-foot by 12-foot pool table has a ball in the corner. Suppose the ball is struck and travels at a 45° angle with a side until it hits a side. The ball then rebounds off that side and travels at a 45° angle until it hits and rebounds off another side. How many times will the ball hit a side before it reaches a corner? The scale drawing shows it will have three hits.

PROBLEMS

Use an accurate drawing or a scale drawing to solve the problems below.

1. Suppose a ball travels as described in the example above on an 8-foot by 10-foot table. How many times will the ball hit a side before it reaches a corner?

2. Answer the same question for a 6-foot by 8-foot pool table.

3. O is the intersection of the perpendicular bisectors of the sides of \( \triangle ABC \), G is the intersection of the medians, and H is the intersection of the altitudes. Observe that O, G, and H are collinear. What fraction of OH is OG? Try several triangles.

4. A pilot in a small plane maintains a constant ground speed of 120 mi/hr. She travels due north for 30 minutes, then northeast for 10 minutes, then southeast for 45 minutes, then southwest for 30 minutes. At that point how long will it take to fly straight home?
Navigation

Traveling in a pleasure boat can be an adventurous way to see the world. But someone on board needs to know about navigation.

Generally speaking, navigation means safely finding your way from one place to another and knowing where you are along the way. Someone who pilots a boat must be constantly aware of the location of the craft, the direction the boat is traveling, and the distance traveled.

Two important tools available to a navigator are a compass and nautical charts. Nautical charts are scaled-down maps of areas of water. They include information about water depth, location of landmarks and ports, and any dangers to navigation in the area.

Two Navigation Techniques

1. Determining the position of a craft by sighting one object.

To determine the position of a craft, it is often useful to find the distance $D$ from the craft to a sighted object. (Fig. 1)
As the boat travels in the direction of $\overrightarrow{P_1P_2}$, a sighting is made of the lighthouse from $P_1$. Later, when the angle of sight has doubled, the sighting is made from $P_2$. The distance, $d$, that the boat travels from $P_1$ to $P_2$ is found using speed and time. The desired distance $D$ is equal to $d$.

Why is this true? What theorem(s) did you use?

2 Determining the position of a craft by sighting three objects.

To determine a boat’s position, a pilot sights three actual objects represented by $A$, $B$, and $C$ on a nautical chart (Fig. 2). Then the pilot measures the angles between the lines of sight. Lines from point $P$ showing these angles are drawn on a sheet of red-tinted plastic. When the plastic is positioned on the chart so that the lines go through $A$, $B$, and $C$, the position of the boat is shown by point $P$.

The method described above will not work, however, when $P$ is on a circle containing $A$, $B$, and $C$. In this picture, for example, the boat could have been at position $R$, $S$, or another location on the circle. What theorem can be used to show that this is true?